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Some properties of the Bézier–Kantorovich type operators

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Abstract

The aim of this paper is to present estimates for the rate of pointwise convergence of the Bézier–Kantorovich modification of the discrete Feller operators in some classes of measurable functions bounded on an interval I, in particular, for functions of bounded pth power variation on I. Our theorems generalize and extend the recent results of Zeng and Piriou (J. Approx. Theory 95(1998) 369; 104(2000) 330) for the kantorovichians of the Bernstein–Bézier operators in the class of functions of bounded variation in the Jordan sense on $[0, 1]$.

Keywords: Discrete Feller operator; Kantorovich and Bézier type operator; Rate of pointwise convergence; Modulus of variation; pth Power variation

1. Introduction

Let $M(I)$ be the class of all measurable real-valued functions bounded on an interval $I \subseteq [0, \infty)$. For $f \in M(I)$, the discrete Feller operator is defined by

$$L_n f(x) := Ef(S_{n,x}/n) = \sum_{j \in J_n} f(j/n) p_{n,j}(x), \tag{1}$$

where $n \in N, x \in I, \{p_{n,j}(x)\colon\ x \in I,\ j \in J_n\}$ is the distribution of the sum $S_{n,x} := X_{1,x} + X_{2,x} + \cdots + X_{n,x}$ and $\{X_{k,x}\colon\ k \in N\}$ is a sequence of independent and identically

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distributed random variables with expectation $EX_{k,x} = x$ for all $k \in N$ and finite variance $\sigma^2(x)$ [2, p. 218]. We assume that the weights $p_{n,j}$ are continuous on I and that the sets J_n are of the form $J_n = N_0 := N \cup \{0\}$ for all $n \in N$, or $J_n = \{0, 1, \ldots, m_n\}$, where $m_n \in N$ and $m_n \leqslant m_{n+1}$ for all $n \in N$. As in [9], we introduce the Bézier basis functions

$$q_{n,k}(x) := \sum_{j \in J_n,\, j \geqslant k} p_{n,j}(x) \quad \text{for } k \in J_n,$$

$q_{n,l}(x) = 0$ for all $l > m_n$ if $J_n = \{0, 1, \ldots, m_n\}$, and $Q^{(\alpha)}_{n,k}(x) := q^{\alpha}_{n,k}(x) - q^{\alpha}_{n,k+1}(x)$, where $\alpha > 0$. The discrete Bézier–type operator $L_{n,\alpha}$ related to (1) and its Kantorovich-type modification $L^*_{n,\alpha}$ are defined by

$$L_{n,\alpha} f(x) := \sum_{k \in J_n} f(k/n) Q^{(\alpha)}_{n,k}(x) \tag{2}$$

and

$$L^*_{n,\alpha} f(x) := \sum_{k \in J_n} |I_{n,k}|^{-1} Q^{(\alpha)}_{n,k}(x) \int_{I_{n,k}} f(t)\, dt, \tag{3}$$

where $I_{n,k} = [a_{n,k}, a_{n,k+1}]$ are the intervals such that $k/n \in I_{n,k}$ for all $k \in J_n$, $I = \bigcup_{k \in J_n} I_{n,k}$ and $|I_{n,k}|$ denotes the measure of $I_{n,k}$.

Recently, Zeng and Piriou [8,9] studied some approximation properties of the special operators $B_{n,\alpha} f$ and $B^*_{n,\alpha} f$ given by (2) and (3), related to the classical Bernstein polynomials $L_n f \equiv B_n f$. In particular, they gave estimates for the rate of pointwise convergence of $B_{n,\alpha} f(x)$ and $B^*_{n,\alpha} f(x)$ for functions f of bounded variation in the Jordan sense on $I = [0, 1]$. Some extensions and generalizations of their results to a general class of operators (2) (with $\alpha \geqslant 1$) can be found in [7]. In this paper we present general estimates for the rate of convergence of $L^*_{n,\alpha} f(x)$ in the case where $f \in M(I)$ and f possesses the one-sided limits $f(x+)$, $f(x-)$ at a fixed point $x \in \operatorname{Int} I$. In our estimates we use the auxiliary function g_x, continuous at x, defined for $t \in I$ by

$$g_x(t) = \begin{cases} f(t) - f(x+) & \text{if } t > x, \\ 0 & \text{if } t = x, \\ f(t) - f(x-) & \text{if } t < x, \end{cases} \tag{4}$$

and the so-called modulus of variation $v_k(g_x; Y)$, $k \in N_0$, of this function on some intervals $Y \subseteq I$. The modulus of variation of a function was first introduced by Lagrange [5]. Some interesting properties of this modulus and its application in the theory of Fourier series were investigated in the papers of Chanturiya (see e.g. [1]). The modulus of variation of a function g on an interval $Y = [c, d]$ is defined as follows: if $k = 0$ then $v_0(g; Y) = 0$; if $k \in N$ then

$$v_k(g; Y) \equiv v_k(g; c, d) := \sup \left\{ \sum_{i=1}^{k} |g(t_i) - g(\tau_i)| \right\},$$

the supremum being taken over all systems $\prod_k$ of k non-overlapping intervals (t_i, τ_i) contained in Y. Clearly, $v_k(g; Y) \leqslant v_{k+1}(g; Y)$ for all $k \in N_0$ and $v_k(g; Z) \leqslant v_k(g; Y)$ for any interval $Z \subset Y$. If $g \in BV_p(Y), p \geqslant 1$, i.e. if g is of bounded pth power variation on Y, then for every $k \in N$,

$$v_k(g; Y) \leqslant k^{1-1/p} V_p(g; Y), \tag{5}$$

where $V_p(g; Y)$ denotes the total pth power variation of g on Y, defined as the upper bound of the set of numbers $(\Sigma_j |g(u_j) - g(s_j)|^p)^{1/p}$ over all finite systems of non-overlapping intervals $(u_j, s_j) \subset Y$.

In order to formulate the main results, we introduce the moments

$$\begin{aligned}\mu^*_{n,\gamma}(x) &:= L^*_{n,1}(|x - \cdot|^\gamma)(x)\\ &= \sum_{k \in J_n} |I_{n,k}|^{-1} p_{n,k}(x) \int_{I_{n,k}} |t - x|^\gamma \, dt,\end{aligned}$$

where $n \in N$, $\gamma > 0$, and we assume that $\mu^*_{n,\gamma}(x) < \infty$ (with some γ occurring below). We restrict ourselves only to the points $x \in I$ at which

$$\sigma^2(x) > 0 \text{ and } \beta(x) := \sum_{j \in J_1} |j - x|^3 p_{1,j}(x) < \infty. \tag{6}$$

(We adopt the convention that $\Sigma_1^0 \cdot = 0$.)

Theorem 1. *Let $f \in M(I)$ and let at a fixed point $x \in \operatorname{Int} I$ the assumptions* (6) *hold and the one-sided limits $f(x+), f(x-)$ exist. Then*

$$\begin{aligned}&|L^*_{n,\alpha} f(x) - 2^{-\alpha} f(x+) - (1 - 2^{-\alpha}) f(x-)|\\ &\quad \leqslant 2((1 + 8n\lambda_n^{(\alpha)}(x)) \left(\sum_{j=1}^{m-1} \frac{1}{j^3} v_j(g_x; Y_x(j/\sqrt{n})) + \frac{1}{m^2} v_m(g_x; Y_x(1)) \right)\\ &\qquad + 2\vartheta_x(1)\lambda_n^{(\alpha)}(x) v_1(g_x; I) + A_\alpha \frac{\Delta(x)}{\sqrt{n}} |f(x+) - f(x-)|\end{aligned}$$

for $n \geqslant n_0(\alpha, x)$, *where* $m = [\sqrt{n}]$, $Y_x(h) = [x - h, x + h] \cap I$, $n_0(\alpha, x) = 1$ *if* $\alpha \geqslant 1$, $n_0(\alpha, x) = (4\beta(x)/\sigma^3(x))^2$ *if* $0 < \alpha < 1$,

$$\lambda_n^{(\alpha)}(x) = \begin{cases} \alpha \mu^*_{n,2}(x) & \text{if } \alpha \geqslant 1,\\ 2^{1-\alpha} (\mu^*_{n,2/\alpha}(x))^\alpha & \text{if } 0 < \alpha < 1, \end{cases} \tag{7}$$

$\vartheta_x(1) = 0$ *if neither of the points* $x-1, x+1$ *belongs to* $\operatorname{Int} I$, $\vartheta_x(1) = 1$ *otherwise*, $A_\alpha = \max\{1, \alpha\}$, *and*

$$\Delta(x) = \frac{5\beta(x)}{2\sigma^3(x)} + \frac{1}{\sqrt{2\pi}\sigma(x)}. \tag{8}$$

Note that for many known operators there exist a non-negative function ψ_α and a positive integer $n(\alpha)$ such that

$$\lambda_n^{(\alpha)}(x) \leqslant \psi_\alpha(x) n^{-1} \text{ for all } x \in I, \ n \geqslant n(\alpha). \tag{9}$$

Theorem 2. *Let* $f \in BV_p(I)$, $p \geqslant 1$, *and let condition* (9) *hold. Then for every* $x \in \operatorname{Int} I$ *at which* (6) *is satisfied and for every* $n \geqslant \max\{n_0(\alpha, x), n(\alpha)\}$ *we have*

$$\begin{aligned} &|L^*_{n,\alpha} f(x) - 2^{-\alpha} f(x+) - (1 - 2^{-\alpha}) f(x-)| \\ &\quad \leqslant \frac{16(1 + 8\psi_\alpha(x))}{(\sqrt{n})^{1+1/p}} \sum_{k=1}^{n} \frac{1}{(\sqrt{k})^{1-1/p}} V_p(g_x; Y_x(1/\sqrt{k})) \\ &\quad + \frac{2}{n} \vartheta_x(1) \psi_\alpha(x) V_p(g_x; I) + A_\alpha \frac{\Delta(x)}{\sqrt{n}} |f(x+) - f(x-)|, \end{aligned}$$

where $Y_x(h), n_0(\alpha, x), \vartheta_x(1), A_\alpha, \Delta(x)$ *are as in Theorem* 1.

If $f \in BV_1(I)$, where $I = [0,1]$, and if $L^*_{n,\alpha} f \equiv B^*_{n,\alpha} f$ are the Bézier–Kantorovich modifications of the Bernstein polynomials, then our Theorem 2 (with $p = 1$) is equivalent to the corresponding results given in [8,9]. Note that the estimate for $B^*_{n,\alpha} f$ following from our Theorem 2 has a slightly different form from that of [9], but in case $\alpha \geqslant 1$ it holds for all $n \in N$ (cf. [9, Theorem 2]).

2. Preliminary results

We first recall that $Q^{(\alpha)}_{n,k}(x) \geqslant 0$ and

$$\sum_{k \in J_n} Q^{(\alpha)}_{n,k}(x) = \left(\sum_{j \in J_n} p_{n,j}(x) \right)^\alpha = 1 \quad \text{for all } \alpha > 0.$$

In view of the obvious inequality $|u^\alpha - v^\alpha| \leqslant \alpha |u - v|$ $(0 \leqslant u, v \leqslant 1, \ \alpha \geqslant 1)$ we have

$$Q^{(\alpha)}_{n,k}(x) \leqslant \alpha(q_{n,k}(x) - q_{n,k+1}(x)) = \alpha p_{n,k}(x) \quad \text{if } \alpha \geqslant 1. \tag{10}$$

Arguing similarly to the proof of Lemma 2 in [8] one can easily verify that

$$\alpha p_{n,k}(x) \leqslant Q^{(\alpha)}_{n,k}(x) \leqslant p^\alpha_{n,k}(x) \quad \text{if } 0 < \alpha < 1. \tag{11}$$

Now, let us represent operator (3) in the form

$$L^*_{n,\alpha} f(x) = \int_I f(t) H^{(\alpha)}_n(x,t)\, dt, \tag{12}$$

where

$$H_n^{(\alpha)}(x,t) = \sum_{k\in J_n} |I_{n,k}|^{-1} Q_{n,k}^{(\alpha)}(x)\chi_{n,k}(t)$$

and $\chi_{n,k}$ are the characteristic functions of the intervals $I_{n,k}$.

Lemma 1. *Let $x, s \in I$. If $s < x$, then*

$$\int_{t\in I, t\leqslant s} H_n^{(\alpha)}(x,t)\,dt \leqslant \frac{A_\alpha}{(x-s)^2}\,\mu_{n,2}^*(x), \tag{13}$$

where $A_\alpha = \max\{1,\alpha\}$. If $x < s$, then

$$\int_{t\in I, t\geqslant s} H_n^{(\alpha)}(x,t)\,dt \leqslant \frac{B_\alpha}{(s-x)^2}\,(\mu_{n,2\gamma}^*(x))^{1/\gamma}, \tag{14}$$

where $B_\alpha = \alpha$ and $\gamma = 1$ if $\alpha \geqslant 1$; $B_\alpha = 2^{1-\alpha}$ and $\gamma = 1/\alpha$ if $0 < \alpha < 1$.

Proof. Clearly, in case $\alpha \geqslant 1$ and $s < x$ we have

$$\int_{t\leqslant s} H_n^{(\alpha)}(x,t)\,dt \leqslant \frac{1}{(x-s)^2}\int_{t\leqslant s}(x-t)^2 H_n^{(\alpha)}(x,t)\,dt \leqslant \frac{\alpha}{(x-s)^2}\,\mu_{n,2}^*(x),$$

by (10). The same bound remains valid for the integral $\int_{t\geqslant s} H_n^{(\alpha)}(x,t)\,dt$ with $\alpha \geqslant 1$ and $x < s$.

Consider now the case where $0 < \alpha < 1$. Choose the integer $l \in J_n$ such that $s \in I_{n,l} = [a_{n,l}, a_{n,l+1}]$. Then, $s = a_{n,l} + \varepsilon|I_{n,l}|$ with some $\varepsilon \in [0,1]$ and, if $s < x$,

$$\begin{aligned}
\int_{t\leqslant s} H_n^{(\alpha)}(x,t)\,dt &= \sum_{k=0}^{l-1} Q_{n,k}^{(\alpha)}(x) + \varepsilon Q_{n,l}^{(\alpha)}(x)\\
&= 1 - (1-\varepsilon)q_{n,l}^{\alpha}(x) - \varepsilon q_{n,l+1}^{\alpha}(x)\\
&\leqslant 1 - (1-\varepsilon)q_{n,l}(x) - \varepsilon q_{n,l+1}(x)\\
&= \sum_{k=0}^{l-1} Q_{n,k}^{(1)}(x) + \varepsilon Q_{n,l}^{(1)}(x) = \int_{t\leqslant s} H_n^{(1)}(x,t)\,dt\\
&\leqslant \frac{1}{(x-s)^2}\,\mu_{n,2}^*(x).
\end{aligned}$$

This completes the proof of (13). If $s > x$, then in view of (11),

$$\begin{aligned}\int_{t \geqslant s} H_n^{(\alpha)}(x,t)\,dt &= Q_{n,l}^{(\alpha)}(x)|I_{n,l}|^{-1}(a_{n,l+1}-s) + \sum_{k \geqslant l+1} Q_{n,k}^{(\alpha)}(x)\\ &= Q_{n,l}^{(\alpha)}(x)|I_{n,l}|^{-1}(a_{n,l+1}-s) + \left(\sum_{k \geqslant l+1} p_{n,k}(x)\right)^{\alpha}\\ &\leqslant p_{n,l}^{\alpha}(x) + \left(\sum_{k \geqslant l+1} p_{n,k}(x)\right)^{\alpha} \leqslant 2^{1-\alpha}\left(\sum_{k \geqslant l} p_{n,k}(x)\right)^{\alpha}\\ &\leqslant \frac{2^{1-\alpha}}{(s-x)^2}\left(\sum_{k \geqslant l} p_{n,k}(x)|I_{n,k}|^{-1}\int_{I_{n,k}} |t-x|^{2/\alpha}\,dt\right)^{\alpha}\\ &\leqslant \frac{2^{1-\alpha}}{(s-x)^2}(\mu_{n,2/\alpha}^{*}(x))^{\alpha}.\end{aligned}$$

Hence inequality (14) follows. □

Lemma 2. *Let $x \in I$ and let $I_x(h) = [x+h, x] \cap I$ if $h < 0$, $I_x(h) = [x, x+h] \cap I$ if $h > 0$. Suppose that g is a function bounded and measurable on $I_x(h)$ and that $g(x) = 0$. Then for all $n \in N$ we have*

$$\begin{aligned}&\left|\int_{I_x(h)} g(t)H_n^{(\alpha)}(x,t)\,dt\right|\\ &\quad\leqslant \left(1 + \frac{8n}{h^2}\lambda_n^{(\alpha)}(x)\right)\left(\sum_{j=1}^{m-1}\frac{1}{j^3}v_j(g; I_x(jh/\sqrt{n})) + \frac{1}{m^2}v_m(g; I_x(h))\right),\end{aligned}$$

where $m = [\sqrt{n}\,]$ and $\lambda_n^{(\alpha)}(x)$ is defined by (7).

Proof. Restricting the proof to $h > 0$, we define the points $t_j := x + jh/\sqrt{n}$ for $j = 1, 2, \ldots, r$, where r is the largest integer such that $t_r \in \operatorname{Int} I_x(h)$ and we denote by t_{r+1} the right end point of the interval $I_x(h)$. Let $T_j := [x, t_j]$ for $j = 1, 2, \ldots, r+1$. Then

$$\begin{aligned}\int_{I_x(h)} g(t)H_n^{(\alpha)}(x,t)\,dt &= \int_x^{t_1} g(t)H_n^{(\alpha)}(x,t)\,dt + \sum_{j=1}^{r} g(t_j)\int_{t_j}^{t_{j+1}} H_n^{(\alpha)}(x,t)\,dt\\ &\quad + \sum_{j=1}^{r}\int_{t_j}^{t_{j+1}} (g(t)-g(t_j))H_n^{(\alpha)}(x,t)\\ &= K_1 + K_2 + K_3, \text{ say.}\end{aligned}$$

Clearly,

$$|K_1| \leqslant \int_x^{t_1} |g(t)-g(x)|H_n^{(\alpha)}(x,t)\,dt \leqslant v_1(g; T_1).$$

Assume $r \geqslant 2$. By the Abel transformation and (14) we have

$$|K_2| \leqslant |g(t_1)| \int_{t_1}^{t_{r+1}} H_n^{(\alpha)}(x,t)\,dt + \sum_{j=1}^{r-1} |g(t_{j+1}) - g(t_j)| \int_{t_{j+1}}^{t_{r+1}} H_n^{(\alpha)}(x,t)\,dt$$
$$\leqslant nh^{-2}\lambda_n^{(\alpha)}(x)\left(|g(t_1)| + \sum_{j=1}^{r-1} |g(t_{j+1}) - g(t_j)|(j+1)^{-2}\right),$$

where $\lambda_n^{(\alpha)}(x) = \alpha\mu_{n,2}^*(x)$ if $\alpha \geqslant 1$, $\lambda_n^{(\alpha)}(x) = 2^{1-\alpha}(\mu_{n,2/\alpha}^*(x))^\alpha$ if $0 < \alpha < 1$.

Using again the Abel transformation and applying some elementary properties of the modulus of variation we obtain

$$|K_2| \leqslant nh^{-2}\lambda_n^{(\alpha)}(x)\left(|g(t_1) - g(x)| + \sum_{j=1}^{r-1} |g(t_{j+1}) - g(t_j)| \frac{1}{r^2}\right.$$
$$\left. + \sum_{j=1}^{r-2} \sum_{i=1}^{j} |g(t_{i+1}) - g(t_i)| \left(\frac{1}{(j+1)^2} - \frac{1}{(j+2)^2}\right)\right)$$
$$\leqslant nh^{-2}\lambda_n^{(\alpha)}(x)\left(v_1(g;T_1) + \frac{1}{r^2} v_{r-1}(g;T_r) + 2\sum_{j=1}^{r-2} \frac{v_j(g;T_{j+1})}{(j+1)^3}\right)$$
$$\leqslant nh^{-2}\lambda_n^{(\alpha)}(x)\left(2\sum_{j=1}^{r-1} \frac{1}{j^3} v_j(g;T_j) + \frac{1}{r^2} v_r(g;T_r)\right).$$

Further, in view of (14),

$$|K_3| \leqslant nh^{-2}\lambda_n^{(\alpha)}(x) \sum_{j=1}^{r} \frac{1}{j^2} v_1(g; t_j, t_{j+1}).$$

Applying the Abel transformation gives

$$|K_3| \leqslant nh^{-2}\lambda_n^{(\alpha)}(x)\left(6\sum_{j=2}^{r} \frac{1}{j^3} v_j(g;T_j) + \frac{1}{r^2} v_r(g;T_{r+1})\right).$$

If $r \geqslant 2$, $r < m$, then

$$\frac{1}{r^2} v_r(g;T_{r+1}) \leqslant 4\sum_{j=r+1}^{m} \frac{1}{j^3} v_j(g;T_j) + \frac{1}{m^2} v_m(g;T_{m+1})$$

and consequently,

$$|K_2 + K_3| \leqslant 8nh^{-2}\lambda_n^{(\alpha)}(x)\left(\sum_{j=1}^{m-1} \frac{1}{j^3} v_j(g;T_j) + \frac{1}{m^2} v_m(g;T_{m+1})\right).$$

Collecting the results and observing that $T_j = I_x(jh/\sqrt{n})$ we get the desired estimate for $h > 0, m \geqslant 2$. A trivial verification shows that it also holds for $m = 1$.

If $h<0$ then the proof runs analogously. In this case we use inequality (13) instead of (14) and we observe that

$$\mu^*_{n,2}(x) \leqslant (\mu^*_{n,2/\alpha}(x))^\alpha \text{ if } 0<\alpha<1,$$

so that right-hand side of (13) can be replaced by $\lambda_n^{(\alpha)}(x)/(x-s)^2$. □

Lemma 3. *Let assumptions* (6) *hold at a fixed* $x \in \operatorname{Int} I$ *and let*

$$\operatorname{sgn}_x^{(\alpha)}(t) := \begin{cases} 2^\alpha - 1 & \text{if } t > x, \\ 0 & \text{if } t = x, \\ -1 & \text{if } t < x. \end{cases} \tag{15}$$

Then

$$|L^*_{n,\alpha} \operatorname{sgn}_x^{(\alpha)}(x)| \leqslant \frac{2^\alpha A_\alpha}{\sqrt{n}} \left(\frac{3\tau\beta(x)}{\sigma^3(x)} + \frac{1}{\sqrt{2\pi}\sigma(x)} \right)$$

for $n \geqslant n_0(\alpha, x)$, *where* $A_\alpha = \max\{1, \alpha\}$, $0<\tau \leqslant 0.8$, $n_0(\alpha, x) = 1$ *if* $\alpha \geqslant 1$ *and* $n_0(\alpha, x) = (4\beta(x)/\sigma^3(x))^2$ *if* $0<\alpha<1$.

Proof. Choose $l \in J_n$ such that $x \in [a_{n,l}, a_{n,l+1}) \equiv I_{n,l} \backslash \{a_{n,l+1}\}$. It is clear that

$$\begin{aligned} L^*_{n,\alpha} \operatorname{sgn}_x^{(\alpha)}(x) &= (2^\alpha - 1) \sum_{k>l} Q^{(\alpha)}_{n,k}(x) - \sum_{k<l} Q^{(\alpha)}_{n,k}(x) \\ &\quad + |I_{n,l}|^{-1} Q^{(\alpha)}_{n,l}(x) \big((2^\alpha - 1)(a_{n,l+1} - x) - (x - a_{n,l}) \big) \\ &= 2^\alpha \sum_{k>l} Q^{(\alpha)}_{n,k}(x) - 1 + 2^\alpha Q^{(\alpha)}_{n,l}(x) |I_{n,l}|^{-1} (a_{n,l+1} - x), \end{aligned}$$

i.e.

$$|L^*_{n,\alpha} \operatorname{sgn}_x^{(\alpha)}(x)| \leqslant 2^\alpha \left| \left(\sum_{j>l} p_{n,j}(x) \right)^\alpha - \frac{1}{2^\alpha} \right| + 2^\alpha Q^{(\alpha)}_{n,l}(x).$$

1° Assume that $\alpha \geqslant 1$. Applying (10) and the inequality $|u^\alpha - v^\alpha| \leqslant \alpha |u - v|$ $(u, v \geqslant 0)$ we get

$$|L^*_{n,\alpha} \operatorname{sgn}_x^{(\alpha)}(x)| \leqslant \alpha 2^\alpha \left(\left| \sum_{j>l} p_{n,j}(x) - \frac{1}{2} \right| + p_{n,l}(x) \right).$$

In view of the Berry–Esséen Theorem [2, p. 515]; [3, p. 93],

$$\left| \sum_{j - nx \leqslant t\sigma(x)\sqrt{n}} p_{n,j}(x) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t \exp(-u^2/2)\, du \right| \leqslant \frac{\tau\beta(x)}{\sqrt{n}\sigma^3(x)}$$

for all $n \in N, t \in R$, where $0 < \tau < 0.8$. From this and from the assumption $k/n \in I_{n,k}$ for all $k \in N$, we have

$$\left|\sum_{j>l} p_{n,j}(x) - \frac{1}{2}\right| = \left|\sum_{j-nx>0} p_{n,j}(x) - \frac{1}{2}\right| \leqslant \frac{\tau\beta(x)}{\sqrt{n}\sigma^3(x)} \tag{16}$$

and

$$p_{n,l}(x) = \sum_{j \leqslant l} p_{n,j}(x) - \sum_{j \leqslant l-1} p_{n,j}(x) \leqslant \frac{2\tau\beta(x)}{\sqrt{n}\sigma^3(x)} + \frac{1}{\sqrt{2\pi}} \int_{s_1}^{s_2} \exp(-u^2/2)\, du,$$

where $s_1 = (l-1-nx)/\sigma(x)\sqrt{n}$, $s_2 = (l-nx)/\sigma(x)\sqrt{n}$. Consequently,

$$p_{n,l}(x) \leqslant \frac{2\tau\beta(x)}{\sqrt{n}\sigma^3(x)} + \frac{1}{\sqrt{2\pi n}\sigma(x)} \quad \text{for all } n \in N.$$

Thus, the desired estimate for $\alpha \geqslant 1$ is established.

2° Consider now the case $0 < \alpha < 1$. By the mean value theorem,

$$\left|\left(\sum_{j>l} p_{n,j}(x)\right)^{\alpha} - \frac{1}{2^{\alpha}}\right| = \alpha(\xi_{n,l}(x))^{\alpha-1}\left|\sum_{j>l} p_{n,j}(x) - \frac{1}{2}\right|,$$

where $\xi_{n,l}(x)$ lies between $\frac{1}{2}$ and $\Sigma_{j>l} p_{n,j}(x)$. In view of (16) we have $\Sigma_{j>l} p_{n,j}(x) > \frac{1}{4}$ for all $n > n_0(x) = (4\beta(x)/\sigma^3(x))^2$. Hence $(\xi_{n,l}(x))^{\alpha-1} \leqslant 4^{1-\alpha}$ for $n > n_0(x)$ and

$$\left|\left(\sum_{j>l} p_{n,j}(x)\right)^{\alpha} - \frac{1}{2^{\alpha}}\right| \leqslant \frac{\tau\beta(x)}{\sqrt{n}\sigma^3(x)}$$

since $\alpha 4^{1-\alpha} \leqslant 1$. Further

$$Q_{n,l}^{(\alpha)}(x) = q_{n,l}^{\alpha}(x) - q_{n,l+1}^{\alpha}(x) = \alpha(\zeta_{n,l}(x))^{\alpha-1} p_{n,l}(x),$$

where $q_{n,l+1}(x) < \zeta_{n,l}(x) < q_{n,l}(x)$. But, in view of (16),

$$\zeta_{n,l}(x) > q_{n,l+1}(x) = \sum_{j \geqslant l+1} p_{n,j}(x) > \frac{1}{4} \quad \text{for } n > n_0(x).$$

Hence

$$Q_{n,l}^{(\alpha)}(x) < \alpha 4^{1-\alpha} p_{n,l}(x) \leqslant \frac{2\tau\beta(x)}{\sqrt{n}\sigma^3(x)} + \frac{1}{\sqrt{2\pi n}\sigma(x)} \quad \text{for } n > n_0(x).$$

Collecting the results we get our estimate for $0 < \alpha < 1$, and the proof is complete. □

3. Proofs of theorems and remarks

Proof of Theorem 1. It is easy to see that under the assumptions of Theorem 1, the function f can be represented in the form

$$f(t) = 2^{-\alpha} f(x+) + (1-2^{-\alpha}) f(x-) + g_x(t) + 2^{-\alpha}(f(x+) - f(x-))\, \mathrm{sgn}_x^{(\alpha)}(t) + (f(x) - 2^{-\alpha} f(x+) - (1-2^{-\alpha}) f(x-))\delta_x(t),$$

where $g_x, \mathrm{sgn}_x^{(\alpha)}$ are defined by (4) and (15), respectively, and $\delta_x(t) = 0$ if $t \neq x, \delta_x(x) = 1$ (see [9, p. 381]). Consequently,

$$L^*_{n,\alpha} f(x) - 2^{-\alpha} f(x+) - (1-2^{-\alpha}) f(x-) = L^*_{n,\alpha} g_x(x) + 2^{-\alpha}(f(x+) - f(x-)) L^*_{n,\alpha}\, \mathrm{sgn}_x^{(\alpha)}(x). \tag{17}$$

Using the representation (12) we can write

$$L^*_{n,\alpha} g_x(x) = \left(\int_{I_x(-1)} + \int_{I_x(1)} \right) g_x(t) H_n^{(\alpha)}(x,t)\, dt + \int_{R_x(1)} g_x(t) H_n^{(\alpha)}(x,t)\, dt,$$

where $I_x(-1) = [x-1, x] \cap I$, $I_x(1) = [x, x+1] \cap I$ and $R_x(1) = I \backslash [x-1, x+1]$. Clearly, $R_x(1)$ is empty if neither of the points $x-1, x+1$ belongs to I. The estimates for the first two integrals are given in Lemma 2, in which we put $g = g_x$ and $h = -1$ or $h = 1$, respectively. Using the obvious inequality

$$v_j(g_x; I_x(-u)) + v_j(g_x; I_x(u)) \leqslant 2 v_j(g_x; Y_x(u)),$$

where $u > 0$, $Y_x(u) = [x-u, x+u] \cap I$, we easily get the estimate for the sum of these two integrals. Since $|g_x(t)| = |g_x(t) - g_x(x)| \leqslant v_1(g_x; I)$, we have

$$\left| \int_{R_x(1)} g_x(t) H_n^{(\alpha)}(x,t)\, dt \right| \leqslant 2\lambda_n^{(\alpha)}(x) v_1(g_x; I),$$

by Lemma 1. Thus the estimate for $|L^*_{n,\alpha} g_x(x)|$ is established. Now, it is enough to apply identity (17) and the estimate of $|L^*_{n,\alpha} \mathrm{sgn}_x^{(\alpha)}(x)|$ given in Lemma 3, and the proof is complete. □

Proof of Theorem 2. If $f \in BV_p(I)$, then in view of (5),

$$\sum_{j=1}^{m-1} \frac{1}{j^3} v_j(g_x; Y_x(j/\sqrt{n})) \leqslant \sum_{j=1}^{m-1} \frac{1}{j^{2+1/p}} V_p(g_x; Y_x(j/\sqrt{n})) \leqslant \frac{2^{2+1/p}}{(\sqrt{n})^{1+1/p}} \int_{1/\sqrt{n}}^{m/\sqrt{n}} \frac{1}{t^{2+1/p}} V_p(g_x; Y_x(t))\, dt$$

$$\leqslant \frac{2^{1+1/p}}{(\sqrt{n})^{1+1/p}} \int_1^n \frac{1}{(\sqrt{s})^{1-1/p}} V_p(g_x;\, Y_x(1/\sqrt{s}))\, ds$$
$$\leqslant \frac{2^{1+1/p}}{(\sqrt{n})^{1+1/p}} \sum_{k=1}^{n} \frac{1}{(\sqrt{k})^{1-1/p}} V_p(g_x;\, Y_x(1/\sqrt{k}))$$

and

$$\frac{1}{m^2} v_m(g_x;\, Y_x(1)) \leqslant \frac{1}{m^{1+1/p}} V_p(g_x;\, Y_x(1)) \leqslant \frac{2^{1+1/p}}{(\sqrt{n})^{1+1/p}} V_p(g_x;\, Y_x(1)).$$

Moreover, $v_1(g_x; I) \leqslant V_p(g_x; I)$. Estimate given in Theorem 2 follows now from Theorem 1 and assumption (9), immediately. □

Remark 1. Let us observe that

$$\sum_{j=1}^{m-1} \frac{1}{j^3} v_j(g_x;\, Y_x(j/\sqrt{n})) \leqslant \sum_{j=1}^{m-1} \frac{1}{j^2} v_1(g_x;\, Y_x(j/\sqrt{n}))$$
$$\leqslant \frac{4}{\sqrt{n}} \int_{1/\sqrt{n}}^{m/\sqrt{n}} t^{-2} v_1(g_x;\, Y_x(t))\, dt$$
$$\leqslant \frac{4}{m} \sum_{k=1}^{m} v_1(g_x;\, Y_x(1/k))$$

and that $v_1(g_x; Y_x(1/k))$ is the oscillation of the function g_x on the interval $Y_x(1/k) = [x - 1/k, x + 1/k] \cap I$. Consequently, in view of the continuity of g_x at x we have

$$\lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^{m} v_1(g_x;\, Y_x(1/k)) = 0.$$

Also

$$\lim_{n \to \infty} \frac{1}{(\sqrt{n})^{1+1/p}} \sum_{k=1}^{n} \frac{1}{(\sqrt{k})^{1-1/p}} V_p(g_x;\, Y_x(1/\sqrt{k})) = 0.$$

Hence, under assumption (9) the right-hand sides of the inequalities given in Theorems 1 and 2 converge to 0 as $n \to \infty$.

Remark 2. A result similar to that of Theorem 2 for functions f of bounded Φ-variation in the Young sense on I can be obtained too (cf. [6, Corollary 1]).

4. An example

Our general results can be applied to concrete operators of the form (3) generated by the known discrete Feller operators (1) such as the Bernstein polynomials, the Szász–Mirakyan operators or the Baskakov ones. It suffices to find the corresponding values of the variance $\sigma^2(x)$ and all parameters in conditions (6) and (9). As an example we consider only the kantorovichians of the Baskakov–Bézier operators. Namely, let

$$U_n f(x) = \sum_{j=0}^{\infty} f(j/n) p_{n,j}(x), \ \ p_{n,j}(x) = \binom{n+j-1}{j} x^j (1+x)^{-n-j}$$

for $x \in I = [0, \infty)$ be the classical Baskakov operators. Denote by $U^*_{n,\alpha} f$ the corresponding operators of the form (3), in which $J_n = N_0$, $I_{n,k} = [k/n, (k+1)/n]$, $|I_{n,k}| = 1/n$ for all $k \in N_0$. As is known, in this case $\sigma^2(x) = x(1+x)$ and

$$\beta(x) = \sum_{j=0}^{\infty} |j-x|^3 p_{1,j}(x) \leqslant \left(\sum_{j=0}^{\infty} (j-x)^2 p_{1,j}(x) \right)^{1/2} \left(\sum_{j=0}^{\infty} (j-x)^4 p_{1,j}(x) \right)^{1/2}$$
$$= x(1+x)(1+9x+9x^2)^{1/2} \leqslant 3x(1+x)^2.$$

Therefore conditions (6) hold at every $x \in (0, \infty)$ and for expression (8) we have the estimate: $\Delta(x) \leqslant 8\sqrt{(1+x)/x}$. Further, it is easy to verify that

$$\mu^*_{n,2}(x) = \frac{x(1+x)}{n} + \frac{1}{3n^2} \leqslant \frac{1 + x(1+x)}{n} \quad \text{for all } n \in N.$$

1° Let $\alpha \geqslant 1$. Then condition (9) is satisfied with $\psi_\alpha(x) = \alpha(1+x+x^2)$ for all $x > 0$ and $n \geqslant n(\alpha)$, where $n(\alpha) = 1$. Consequently, for operators $U^*_{n,\alpha} f$ one can deduce estimates as in Theorems 1 and 2. We will formulate only the result following from Theorem 2.

Corollary 1. *If $f \in BV_p(I)$, where $I = [0, \infty), p \geqslant 1$, and if $\alpha \geqslant 1$ then for all $x > 0$ and $n \in N$ we have*

$$|U^*_{n,\alpha} f(x) - 2^{-\alpha} f(x+) - (1 - 2^{-\alpha}) f(x-)|$$
$$\leqslant \frac{16(1 + 8\alpha(1+x+x^2))}{(\sqrt{n})^{1+1/p}} \sum_{k=1}^{n} \frac{1}{(\sqrt{k})^{1-1/p}} V_p(g_x; Y_x(1/\sqrt{k}))$$
$$+ \frac{2\alpha(1+x+x^2)}{n} V_p(g_x; I) + \frac{8\alpha}{\sqrt{n}} \sqrt{\frac{1+x}{x}} |f(x+) - f(x-)|,$$

where $Y_x(1/\sqrt{k}) = [x - 1/\sqrt{k}, x + 1/\sqrt{k}] \cap [0, \infty)$.

2° Let $0 < \alpha < 1$. In order to verify condition (9), we need to estimate the function $(\mu^*_{n,2/\alpha}(x))^\alpha$. Write $l = 2/\alpha$ and denote by $[l]$ the greatest integer not exceeding l. As in

[8, Lemma 6] choose the numbers:

$$p = \frac{2[l]}{2[l] + 2 - l}, \quad p' = \frac{2[l]}{l - 2}, \quad r = \frac{2}{p}, \quad s = \frac{2v}{p'}, \quad v = [l] + 1.$$

Clearly, $l > 2$, $p > 1, p' > 1, 1/p + 1/p' = 1$ and $l = r + s$. Applying twice the Hölder inequality (first for integrals, next for sums) we obtain

$$\int_{I_{n,k}} |t-x|^l \, dt \leqslant \left(\int_{I_{n,k}} |t-x|^{rp} \, dt \right)^{1/p} \left(\int_{I_{n,k}} |t-x|^{sp'} \, dt \right)^{1/p'}$$

and then

$$\mu^*_{n,l}(x) \leqslant (\mu^*_{n,2}(x))^{1/p} (\mu^*_{n,2v}(x))^{1/p'}.$$

Since $v \in N$, we have

$$\mu^*_{n,2v}(x) = n \sum_{k=0}^{\infty} p_{n,k}(x) \int_{k/n}^{(k+1)/n} (t-x)^{2v} \, dt = \frac{1}{(2v+1)n^{2v}} \sum_{i=0}^{2v} \binom{2v+1}{i} T_{n,i}(x),$$

where

$$T_{n,i}(x) = \sum_{k=0}^{\infty} p_{n,k}(x)(k - nx)^i.$$

From the representation of $T_{n,i}(x)$ given in [4, Corollary 3.7] it may be concluded that

$$\mu^*_{n,2v}(x) \leqslant \frac{c(v)}{n^v} (1+x) \sum_{j=0}^{v} (x(1+x))^j \text{ for all } x \in I, n \geqslant 1,$$

where $c(v)$ is a positive constant depending only on v. Therefore

$$\mu^*_{n,l}(x) \leqslant c(v)(1 + x(1+x))^{1/p} \left((1+x) \sum_{j=0}^{v} (x(1+x))^j \right)^{1/p'} n^{-(1/p + v/p')}$$

$$\leqslant c(v) \sum_{j=0}^{v} x^j (1+x)^{j+1} n^{-1/\alpha}.$$

This means that $\lambda_n^{(\alpha)}(x) = 2^{1-\alpha}(\mu^*_{n,2/\alpha}(x))^\alpha \leqslant \psi_\alpha(x) n^{-1}$ for all $n \in N$, where

$$\psi_\alpha(x) = \kappa(\alpha) \left(\sum_{j=0}^{[2/\alpha]+1} x^j (1+x)^{j+1} \right)^\alpha,$$

and $\kappa(\alpha)$ is a positive constant depending only on α. Consequently, if $0 < \alpha < 1$, then estimates given in Theorems 1 and 2 hold for operators $U^*_{n,\alpha} f$ with the above values of $\lambda_n^{(\alpha)}(x), \psi_\alpha(x)$ and with $n(\alpha) = 1$, $n_0(\alpha, x) \leqslant 144(1+x)/x$, $\Delta(x) \leqslant 8\sqrt{(1+x)/x}$.

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